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LETTER TO THE EDITOR

Random manifolds in non-linear resistor networks: applications to varistors and superconductors

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Abstract

We show that current localization in polycrystalline varistors occurs on paths which are usually in the universality class of the directed polymer in a random medium. We also show that, in ceramic superconductors, voltage localizes on a surface which maps to an Ising domain wall. The emergence of these manifolds is explained and their structure is illustrated using direct solution of non-linear resistor networks.

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(Some figures in this article are in colour only in the electronic version)

Low energy paths and surfaces in random systems are of broad importance in statistical mechanics and in materials theory. Examples include the morphology of flux lines in the presence of random pinning [1], magnetic domain walls in random-bond Ising magnets [2, 3], dielectric breakdown paths [4, 5] and fracture surfaces [6]. The directed polymer in a random medium has received an enormous amount of interest, both due to its intrinsic importance and due to its relation with growth processes [7]. Domain walls in random-bond Ising models have also received a good deal of attention. In fact the directed polymer in a random medium was first used to model domain walls in two-dimensional random-bond magnets [2]. More recently it has been realized that random path and surface problems map to classic problems in computer science [8, 9]. In particular, the *shortest-path problem* is related to the directed polymer in a random media [10] and the *minimum cut problem* is related to domain walls in random Ising magnets [11]. The shortest path and minimum cut problems have polynomial time algorithms, and this has enabled physicists to solve path problems *with overhangs* (i.e. not just the directed case) [10] and domain wall problems in three dimensions [11], both of which were previously considered difficult. Many more connections between statistical physics and solvable combinatorial optimization problems have recently proven profitable [12]. Here we

show that the shortest path and minimum cut problems also emerge in the study of the onset of current or voltage in non-linear resistor networks and in their applications to varistors and superconductors. This enables a detailed characterization of the manifolds on which current or voltage localize near critical thresholds.

Non-linear flow in random networks is germane to problems ranging from non-Newtonian fluid flow in porous media [13], to the onset of flow in varistors [14–17], and to the onset of voltage in grain-boundary limited superconductors [18–20]. The emergence of special flow paths in the fluid and varistor cases is well established and network models with realistic non-linear flow laws have been constructed [13–15]. Recently lattice models have also been used to model the behaviour of polycrystalline superconductors [20]. The networks which model these materials have highly non-linear current–voltage (V–I) (or pressure–flow) behaviour on each bond of a graph. In the case of diodes or varistors each bond has a critical voltage, v_c^k, which varies from bond to bond [14, 15, 21, 22]. The onset of current flow occurs on special paths and the identification and characterization of these paths is a central issue in the analysis of varistors. In the superconductor case, each bond has a local critical current, i_c^k, below which the bond carries current *but has zero voltage*. Above this threshold current, a local voltage must be applied to the bond in order to increase the current further. The onset of voltage occurs on a surface [20] which we show maps to a minimum cut through the network.

Firstly we define the shortest path and minimum cut problems [8, 9]. These problems (and the algorithms which solve them) are very basic to algorithmic systems in computer science. They are defined on a graph where each edge is assigned a cost (shortest-path problem) or a capacity (minimum cut problem). Consider then a graph composed of nodes and edges, where each edge has a cost, c_k. The shortest path between any two sites, s and t, in the graph is simply the path on which the sum of the edge costs

$$C_{st} = \min_P \left(\sum_{k \epsilon P} c_k \right), \tag{1}$$

on the path is *smallest*. In the language of physics the shortest path is the lowest energy path. It is clear that this is the same as the problem of a directed polymer in a random media, provided the path is forced to be directed. However, if overhangs are allowed this problem can also be solved efficiently using, for example, Dijkstra's method. It turns out that overhangs do not change the universality class of the problem unless disorder is strong [23, 24].

The minimum cut problem [8, 9] is defined on a graph consisting of nodes and edges in which each edge has a flow *capacity*, u_k. This is the maximum amount of flow that the edge can carry. There is no energy cost in the computer science definition of this problem, there is just the limiting capacity, u_k (however, in the end some of the physics applications associate the capacity with an energy). Now consider the amount of flow that can be carried between two sites, s and t. A key theorem states that the maximum flow between s and t is equivalent to the capacity of the minimum cut between s and t [8, 9]. The minimum cut is the lowest capacity surface of separation in which s is on one side of the cut and t is on the other. The capacity of the minimum cut, U_{st}, is the sum of the capacities on this surface of separation, S:

$$U_{st} = \min_S \left(\sum_{k \epsilon S} u_k \right). \tag{2}$$

If we associate the bond capacities with the exchange constants in a random-bond Ising magnet, then the minimum cut maps to the lowest energy domain wall. Somewhat surprisingly, the minimum cut can be found efficiently (in almost linear time in the number of edges) and this has enabled detailed study of domain walls in three dimensions [11, 30].

Now we consider in more detail the emergence of the shortest path and the minimum cut in non-linear random resistor networks. To be concrete, we consider square networks where each

bond has a *monotonic* (single-valued) non-linear V–I behaviour. The case of current onset at a critical voltage is applicable to varistors [16] (figure 1) and diodes and also to Bingham plastics [13] (replace voltage by pressure and current flow by fluid flow so that flow initiates at a critical pressure). The case of a voltage onset at a critical current is applicable to ideal Josephson junctions (figure 2) and also to ideal flux-flow laws [20]. Ideal varistor behaviour is mathematically described by $i(v) = (i_0 + (v - v_c)/r)\theta(v - v_c)$, while ideal Josephson junction or flux flow behaviour is described by $v(i) = (v_0 + (i - i_c)/g)\theta(i - i_c)$, where $\theta(x)$ is the step function which is zero for $x < 0$ and one for $x \geqslant 0$. Finding the current or fluid flow through these non-linear networks seems difficult due to the non-smooth behaviour at the critical current or voltage. However, a variety of regularization methods are available to handle these difficulties. The simplest is to replace the step function by $\theta(x) = x/(\xi^2 + x^2)^{1/2}$ and to take the limit $\xi \to 0$. This form has nice analytic properties which assist in the numerical analysis, so we have used this regularization in our codes. A key observation is that these problems have a related cost function defined by cost $= \int_0^i v(i')\,\mathrm{d}i'$, which is *convex* provided the V–I law on each bond is monotonically increasing. With suitable regularization this is true and solving these non-linear resistor networks then reduces to a convex optimization problem with the constraint of flow conservation at each node. We have used this framework to develop efficient codes for this class of problem. Here we discuss applications to varistors and superconductors.

In the varistor case (see figure 1) each bond has a random onset voltage, v_c^k, and the whole network has the onset voltage, V_c. At V_c, current only flows on the shortest path through the network, as we confirmed numerically by finding the path on which $V_c = \sum_{k \epsilon P} v_c^k$ is minimal (using Dijkstra's method). Voltage localization in the superconductor case is illustrated in figure 2. In this case each bond has a random critical current, i_c^k, and the whole network has the critical current, I_c. We solved the full flow equations directly and then confirmed that, at I_c, voltage localizes on the *minimum cut*, i.e. the surface on which $I_c = \sum_{k \epsilon S} i_c^k$ is minimal.

The easiest way to prove that random manifolds emerge at the macroscopic thresholds, I_c and V_c, is by using the cost function, $\int_0^i v(i')\,\mathrm{d}i'$. From this cost function, it is evident that if either the current or voltage can be kept to zero, then the cost itself is zero. In the case of a varistor, as the external voltage is increased, the network distributes the voltage drops so as to keep the current at zero. It succeeds in doing this until it is no longer possible and the first time this occurs is on the shortest path, occurring at the critical voltage, V_c. This argument applies to *any cost function which has strictly zero current up to threshold*. This means that the behaviour of the local $V - -I$ characteristic after threshold has *no influence* on the value of V_c. In a similar manner, in the superconductor case, a large network attempts to keep the voltage at zero in all of the bonds until it is impossible to do so. As the applied current is increased, the current is distributed so as to keep the currents in all of the bonds below their threshold value, again in a very cooperative manner. The first applied current, I_c, at which voltage appears is determined by when a surface of bonds all have reached their local critical currents. This surface is the minimum cut.

To make contact with experiment, note that varistors are materials which are insulating below a critical electric field, E_c, after which they become highly conducting [14–16]. The physics of their operation is understood to originate in the grain boundaries of polycrystalline ceramics. The interior of the grains of typical varistors are conducting, but the grain boundaries are insulating at low voltage (below about 3 V). The onset of current flow occurs when the applied voltage is large enough to induce a path of grain boundaries to become conducting. This physics is naturally encapsulated in lattice models in which a node represents a grain and a bond represents a grain boundary [14–16]. However, the grains have varying sizes and the grain boundaries have varying properties, with some boundaries having higher onset voltages than

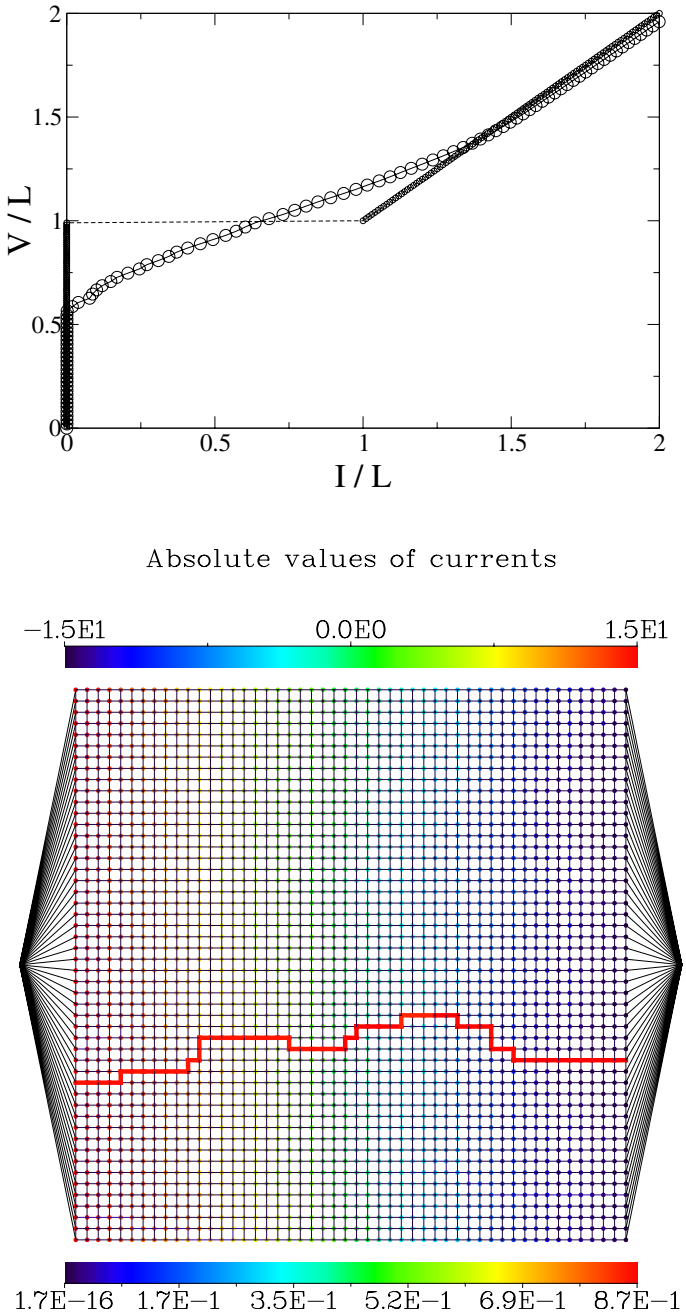

Figure 1. Behaviour of a 50×50 varistor network, with the bonds having onset voltages uniformly distributed on the interval [0, 2], asymptotic resistance 1 Ω, and with the external voltage applied in the horizontal direction. The behaviour of the average bond is the abrupt curve (smaller circles) in the top figure, while the behaviour of the network is the smoother curve in the top figure (bigger circles). The localization of current on the shortest path is illustrated in the lower figure. This is the current pattern at $V/L = 0.59$. At higher voltages further filamentary current paths emerge.

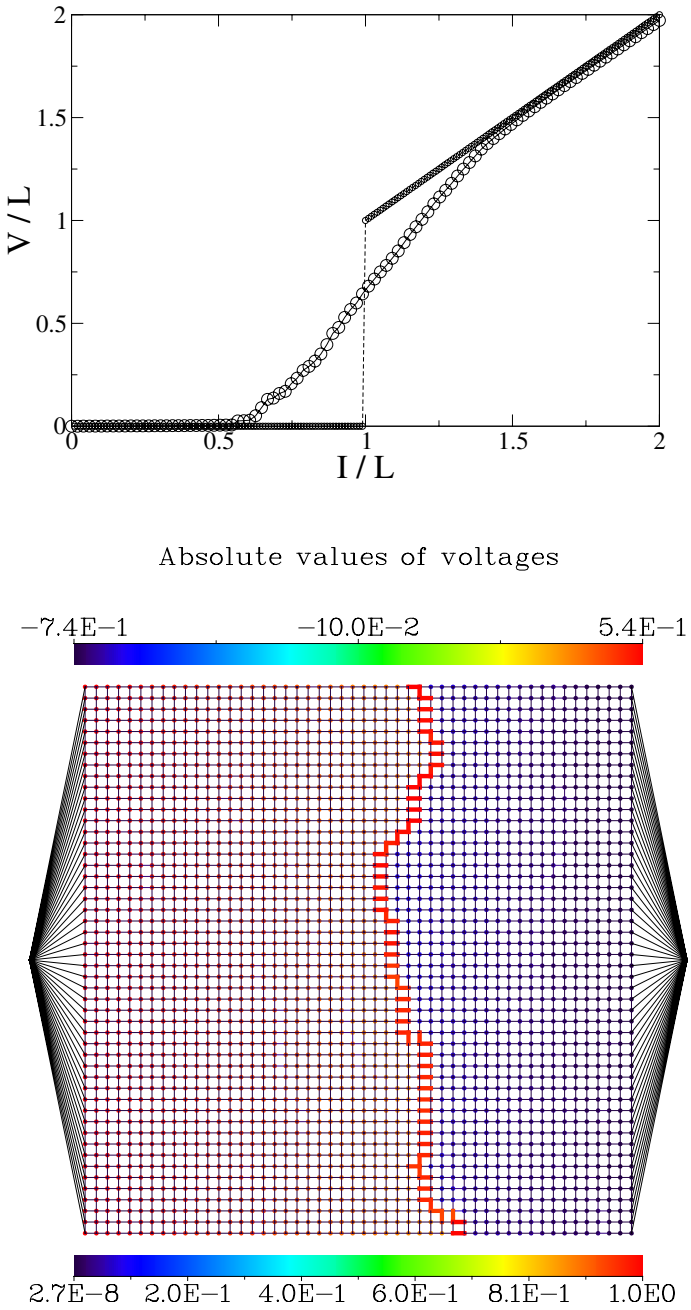

Figure 2. Behaviour of a 50×50 Josephson junction network, with the bonds having onset currents uniformly distributed on the interval [0, 2] and asymptotic resistance 1 Ω, and with the external current injected in the horizontal direction. The behaviour of the average bond is the abrupt curve (smaller circles) in the top figure, while the behaviour of the network is the smoother curve in the top figure (bigger circles). The localization of voltage on the minimum cut is illustrated in the lower figure. This is the voltage pattern at $I/L = 0.58$. At higher currents further sheet-like voltage surfaces emerge.

others. That is, we must study disordered networks. These networks are also highly non-linear as has been demonstrated by studies of flow onset across individual grain boundaries [16]. The linear dimension, L, of the lattice models is related to the size of real varistors by $L = l/g$, where l is the sample size and g is the grain size. Considerable progress has been made using numerical simulations which have provided good agreement with experiment [14–16]. Here we make the connection with the statistical physics of random paths.

As discussed above, network models of polycrystalline varistors have a current onset given by the minimum over all paths of $V_c = \sum_{k \epsilon P} v_c^k$. This is just the shortest-path problem of equation (1), where the voltage threshold on a bond replaces the cost of that bond. The shortest-path problem is usually in the universality class of the directed polymer in a random medium (DPRM) [10] (except for strong disorder [23, 24]) so that V_c behaves like the *energy* of a DPRM, i.e.

$$V_c = a_1 L + a_2 L^{\theta}, \tag{3}$$

where a_1 and a_2 are independent of sample size and the exponent θ is universal. Its value is known to be exactly $1/3$ [2] for paths through two-dimensional systems and 0.248 ± 0.004 for paths through three-dimensional media [27]. A key feature of (3) is that the threshold electric field of the lattice $E_c = V_c/L$ is *size independent*, in contrast to the size effects produced by rare-fluctuation theories of electrical and dielectric failure [4, 5, 28], where E_c approaches zero logarithmically in the large lattice limit. However, the key difference is that, in dielectric breakdown, local regions irreversibly make the transition to the conducting state, so that no voltage is required to maintain these regions in the conducting state after failure. In contrast, high quality varistors are reversible so that there is a steady state current at a fixed applied voltage. The path on which current flows is usually self-affine and has length, L_p, given by [2, 27], $L_p = b_1 L + b_2 L^{\zeta}$, where b_1 and b_2 are dependent on the disorder distribution, but the exponent ζ is universal. Its value is known to be exactly $2/3$ [2] for paths through two-dimensional systems and 0.62 ± 0.01 [27] for paths through three-dimensional media. The exponents θ and ζ are related by $\theta = 2\zeta - 1$, which is consistent with the numerical results quoted above. The difference in voltage between the lowest threshold path and the onset voltage of the next filamentary current path scales in the same way as the energy gap in the DPRM problem, which decreases logarithmically with increasing sample size [25]. In the strong disorder limit, the paths become highly tortuous and are no longer in the DPRM class [23, 24, 26].

In the case of ceramic superconductors the low-angle grain boundaries have much higher critical currents than the high-angle grain boundaries [29]. As in the varistor case, lattice models take the grain centres to be nodes and the links between grains to represent the grain boundaries [18, 20]. Grain boundaries may have a flux flow character or a Josephson junction character or be resistive. If a grain boundary is resistive, its critical current is zero. As demonstrated above, the onset of voltage occurs on the minimum cut through such a network and this minimum cut is related to domain walls in Ising magnets [2, 3, 11, 30]. The critical current of these networks then behaves in the same way as the energy of Ising domain walls, i.e.

$$I_c = c_1 L^{d-1} + c_2 L^{\theta} \tag{4}$$

where d is the spatial dimension, c_1 and c_2 are independent of sample size and $\theta = 1/3$ in two dimensions and $\theta = 0.82 \pm 0.02$ in three dimensions. There is thus a sample size independent critical current density in these networks, in contrast to the size effect which occurs in fuse networks [4, 28]. Again this difference is due to the fact that these networks must maintain a current of at least i_c^k on the kth bond in order for a voltage to appear there. The surface on which voltage localizes at I_c (i.e. the minimum cut) is in the universality class of domain walls in the random-bond Ising model and so it is self-affine and has asymptotic roughness, w, given by $w = c_3 L^{\zeta}$,

where $\zeta = 2/3$ in two dimensions and $\zeta \sim 0.41 \pm 0.01$ in three dimensions [3, 11, 30]. The exponents θ and ζ are related to each other by the scaling relation $\theta = 2\zeta + d - 3$.

We have demonstrated that current or voltage localize on special manifolds in non-linear networks. As particular examples, we showed that flow in a varistor begins on a path which is equivalent to a shortest path through the random medium, and in superconductors, voltage onset occurs on a minimum cut through the network. The macroscopic current or voltage thresholds are then equivalent to the *energy* of a random manifold. Since these macroscopic thresholds only depend on the *local current or voltage thresholds* (and not on the local $V - -I$ characteristics after threshold) the manifolds on which current or voltage localize in varistors and superconductors are described by the universal directed polymer or random surface exponents.

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